

# A VERSION OF THE VOLUME CONJECTURE

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**ABSTRACT.** We propose a version of the volume conjecture that would relate a certain limit of the colored Jones polynomials of a knot to the volume function defined by a representation of the fundamental group of the knot complement to the special linear group of degree two over complex numbers. We also confirm the conjecture for the figure-eight knot and torus knots. This version is different from S. Gukov's because of a choice of polarization.

## 1. INTRODUCTION

For a knot  $K$  in the three-sphere  $S^3$ , one can define the colored Jones polynomial  $J_N(K; t)$  as the quantum invariant corresponding to the  $N$ -dimensional irreducible representation of the Lie algebra  $sl(2; \mathbb{C})$  [5, 9] (see also [8]).

The volume conjecture [6, 14] states that the limit of  $\log(J_N(K; \exp(2\pi\sqrt{-1}))) / N$  would determine the simplicial volume of the knot complement  $S^3 \setminus K$ . In [15] Y. Yokota and the author proved that for the figure-eight knot  $E$  and a complex number  $r$  the limit  $\log(J_N(E; \exp(2\pi r\sqrt{-1}))) / N$  also exists and defines the volume for the three-manifold obtained from  $S^3$  by certain Dehn surgery if  $r$  is close to 1.

In this paper we will show that a similar phenomenon appears for torus knots, which are not hyperbolic. We also propose a version of the volume conjecture which relates the limit of  $\log(J_N(K; \exp(2\pi r\sqrt{-1}))) / N$  to the volume function corresponding to a representation of  $\pi_1(S^3 \setminus K)$  to  $SL(2; \mathbb{C})$  such that ratio of the eigenvalues of its image of the meridian is  $\exp(2\pi r\sqrt{-1})$ . Our version of generalization of the volume conjecture is different from S. Gukov's [4] due to a choice of polarization.

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## 2. A CONJECTURE

Let  $K$  be a knot in  $S^3$ . Denote by  $J_N(K; q)$  the colored Jones polynomial associated to the  $N$ -dimensional irreducible representation of the Lie algebra  $sl(2, \mathbb{C})$  [9]. We normalize it so that  $J_2(K; q)$  is the Jones polynomial [5] and that  $J_N(U; q) = 1$ , where  $U$  is the unknot.

We propose the following conjecture as a version of the volume conjecture.

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**Conjecture 2.1** (Parameterized Volume Conjecture). *There exists an open subset  $\mathcal{O}_K$  of  $\mathbb{C}$  such that for any  $u \in \mathcal{O}_K$  the following limit exists:*

$$(2.1) \quad \lim_{N \rightarrow \infty} \frac{\log J_N(K; \exp((u + 2\pi\sqrt{-1})/N))}{N}.$$

Moreover the function of  $u$

$$(2.2) \quad H(K; u) := (u + 2\pi\sqrt{-1}) \lim_{N \rightarrow \infty} \frac{\log J_N(E; \exp((u + 2\pi\sqrt{-1})/N))}{N}$$

is analytic on  $\mathcal{O}_K$ . If we put

$$v_K(u) := 2 \frac{dH(K; u)}{du} - 2\pi\sqrt{-1},$$

then the following formula holds:

$$(2.3) \quad V(K; u) = \operatorname{Im}(H(K; u)) - \pi \operatorname{Re}(u) - \frac{1}{2} \operatorname{Re}(u) \operatorname{Im}(v_K(u)),$$

where  $V(K; u)$  is the volume function corresponding to the representation from  $\pi_1(S^3 \setminus K)$  to  $SL(2; \mathbb{C})$  sending the meridian and the longitude to the elements the ratios of whose eigenvalues are  $\exp(u)$  and  $\exp(v_K(u))$  respectively [1, § 4.5].

*Remark 2.2.* If  $u$  is parameterized by a real number  $t$ , then  $V(K; u(t))$  satisfies the following differential equation from Schläfli's formula:

$$(2.4) \quad \frac{dV(K; u(t))}{dt} = -\frac{1}{2} \left( \operatorname{Re}(u(t)) \frac{d \operatorname{Im}(v_K(u(t)))}{dt} - \operatorname{Re}(v_K(u(t))) \frac{d \operatorname{Im}(u(t))}{dt} \right).$$

See [16, § 5] and [1, § 4.5]. Note that we use the same convention for the meridian/longitude pair as in [16], which is different from that in [1] and [4]. Note also that the right hand side of the equation in the last line of Page 62 of [1] should be multiplied by four (see [4, (5.6)]).

*Remark 2.3.* If there exist coprime integers  $p$  and  $q$  satisfying  $pu + qv_K(u) = 2\pi\sqrt{-1}$ , then  $u$  would define the  $(p, q)$ -Dehn surgery along  $K$  [17]. If this is a hyperbolic manifold,  $V(K; u)$  is its hyperbolic volume.

*Remark 2.4.* The open set  $\mathcal{O}_K$  may not contain 0. Therefore Conjecture 2.1 is not a generalization of the volume conjecture. (Recall that the volume conjecture [6, 14] states that when  $u = 0$ , then the limit (2.1) gives the simplicial volume of the knot complement.) In fact the case where  $K$  is a torus knot, the limit (2.1) is not continuous at 0 [7, 12]. See also [3, Proposition B.2].

*Remark 2.5.* S. Garoufalidis and T. Le proved that if  $u$  is close enough to  $-2\pi\sqrt{-1}$ , then  $J_N(K; \exp((u + 2\pi\sqrt{-1})/N))$  converges to  $1/\Delta(K; \exp(u + 2\pi\sqrt{-1}))$ , where  $\Delta(K; t)$  is the Alexander polynomial of  $K$  [2]. (See also [11] for the figure-eight knot.) In this case the right hand side of (2.1) vanishes and so we have  $H(K; u) = 0$  and  $v_K(u) = -2\pi\sqrt{-1}$ . Therefore from (2.3) the volume function  $V(K; u)$  vanishes. This corresponds to the case where  $u$  defines an abelian representation, whose volume function is zero. See [3, Appendix B]. So we exclude such a case in Conjecture 2.1.

Note that the conjecture above is proved for the figure-eight knot by Yokota and the author [15]. In fact we proved [15, Corollary 2.4] that for the figure-eight knot  $E$

$$V(E; u) = \operatorname{Im}(H(E; u)) - \pi \operatorname{Re}(u) - \frac{1}{4} \operatorname{Im}(u v_E(u)) - \frac{\pi}{2} \operatorname{length}(\gamma),$$

where  $\text{length}(\gamma)$  is the length of the geodesic  $\gamma$  added to complete the incomplete hyperbolic structure of  $S^3 \setminus E$  corresponding to  $u$ . But since  $\text{length}(\gamma) = -\text{Im}(u \overline{v_E(u)})/(2\pi)$  from [16, (34)], we have

$$\begin{aligned} V(E; u) &= \text{Im}(H(E; u)) - \pi \text{Re}(u) - \frac{1}{4} \text{Im}(u v_E(u)) + \frac{1}{4} \text{Im}(u \overline{v_E(u)}) \\ &= \text{Im}(H(E; u)) - \pi \text{Re}(u) - \frac{1}{2} \text{Re}(u) \text{Im}(v_E(u)) \end{aligned}$$

as required.

See also [13, (4.1)]. (The sign of  $\pi \text{Re } u$  in [13, (4.1)] should be negative because the author used a wrong definition of the function  $H$  in the old version of [15].)

Note also that Gukov uses a different *polarization* in his generalization of the volume conjecture [4, (5.12)]. It agrees with Conjecture 2.1 when  $\text{Re}(u) = 0$ . The difference for  $\text{Re}(u) \neq 0$  can be explained by a choice of polarization. Details will be described in a forthcoming paper.

### 3. PROOF

In this section we prove Conjecture 2.1 for torus knots. (To be honest, (2.3) is proved only up to a constant, that is, we prove (2.4) instead.)

Let  $T(a, b)$  be the torus knot of type  $(a, b)$ , where  $a$  and  $b$  are coprime integers with  $a > 1$  and  $b > 1$ . Then the author proved in [12, Theorem 1.1] the following theorem.

**Theorem 3.1** ([12]). *Suppose that  $|r| > 1/(ab)$ ,  $\text{Re } r > 0$  and  $\text{Im } r > 0$ , then*

$$\lim_{N \rightarrow \infty} \frac{\log J_N(T(a, b); \exp(2\pi r \sqrt{-1}/N))}{N} = \left(1 - \frac{1}{2abr} - \frac{abr}{2}\right) \pi \sqrt{-1}.$$

Therefore the functions  $H(T(p, q); u)$  and  $v_{T(a, b)}(u)$  in Conjecture 2.1 are defined as follows:

$$\begin{aligned} H(T(p, q); u) &= (u + 2\pi\sqrt{-1}) \left(1 - \frac{1}{2ab \left(1 + \frac{u}{2\pi\sqrt{-1}}\right)} - \frac{ab \left(1 + \frac{u}{2\pi\sqrt{-1}}\right)}{2}\right) \pi \sqrt{-1} \\ &= \frac{-(ab(u + 2\pi\sqrt{-1}) - 2\pi\sqrt{-1})^2}{4ab} \end{aligned}$$

and

$$v_{T(a, b)}(u) = 2 \frac{d H(T(a, b); u)}{d u} - 2\pi\sqrt{-1} = -ab(u + 2\pi\sqrt{-1})$$

for  $u$  with  $|u + 2\pi\sqrt{-1}| > 2\pi/(ab)$ ,  $\text{Re}(u) < 0$  and  $\text{Im}(u) > -2\pi$ .

So the volume function  $V(T(a, b); u)$  is

$$\begin{aligned} V(T(a, b); u) &= \text{Im}(H(T(a, b); u)) - \pi \text{Re}(u) - \frac{1}{2} \text{Re}(u) \text{Im}(v_{T(a, b)}(u)) \\ &= -\frac{1}{4} ab \text{Im}(u^2) - ab\pi \text{Re}(u) + \pi \text{Re}(u) \\ &\quad - \pi \text{Re}(u) - \frac{1}{2} \text{Re}(u)(-ab \text{Im}(u) - 2ab\pi) \\ &= 0. \end{aligned}$$

On the other hand the right hand side of (2.4) equals

$$-\frac{1}{2} \left( \text{Re}(u(t))(-ab) \frac{d \text{Im}(u(t))}{dt} + ab \text{Re}(u(t)) \frac{d \text{Im}(u(t))}{dt} \right) = 0,$$

which proves (2.4).

This confirms Conjecture 2.1 for the torus knot  $T(a, b)$ .

*Remark 3.2.* The volume function for a torus knot would be zero. See [3, Appendix B].

*Remark 3.3.* Note that the pair  $(u, v_{T(a,b)}(u))$  satisfies the equality:

$$(-1) \times u + \left( \frac{-1}{ab} \right) \times v_{T(a,b)}(u) = 2\pi\sqrt{-1}.$$

So the corresponding geometric object would be the generalized Dehn surgery along the torus knot  $T(a, b)$  with invariant  $(1, 1/(ab))$ , or the  $ab$ -Dehn surgery with cone-angle  $2ab\pi$  [17, Chapter 4, § 4.5] (see also Remark 2.3). It would be interesting that the  $ab$ -Dehn surgery along the torus knot  $T(a, b)$  is reducible [10].

#### 4. COMMENTS

In [4, (5.29)], Gukov proposed the following conjecture.

**Conjecture 4.1.** [4] *Let  $K$  be a knot in the three-sphere. For  $a \in \mathbb{C} \setminus \mathbb{Q}$ , define the function  $l(a)$  as follows:*

$$(4.1) \quad l(a) := -\frac{d}{da} \left\{ a \lim_{N \rightarrow \infty} \frac{\log J_N(K; \exp(2\pi a \sqrt{-1}/N))}{N} \right\}.$$

*Then the pair  $(\exp(l(a)), -\exp(\pi a \sqrt{-1}))$  is a zero of the  $A$ -polynomial of  $K$  introduced in [1].*

Using our parameterization  $u := 2\pi a \sqrt{-1} - 2\pi \sqrt{-1}$ , we have  $l(a) = -v_K(u)/2 - \pi \sqrt{-1}$ . Then Conjecture 4.1 states that the pair  $(-\exp(-v_K(u)/2), \exp(u/2))$  is a zero of the  $A$ -polynomial.

In the case of the figure-eight knot, we can prove this. For the torus knot  $T(a, b)$ , either  $(-\exp(-v_K(u)/2), \exp(u/2))$  or  $(-\exp(v_K(u)/2), \exp(u/2))$  is a zero of the  $A$ -polynomial. (See, for example, [18, Example 4.1] for the  $A$ -polynomials of torus knots.) This would depend on how we choose a meridian/longitude pair.

*Remark 4.2.* If  $|a|$  is small enough the right hand side of (4.1) vanishes [3]. This corresponds to the  $(1-1)$ -factor of the  $A$ -polynomial [1, 2.5]. See Remark 2.5.

The function  $H(K; u)$  defined by (2.2) is a kind of potential function introduced in [16, Theorem 3]. (To be more precise,  $\Phi(u) = 4H(K; u) - 4\pi u \sqrt{-1}$ , where  $\Phi(u)$  is Neumann–Zagier’s potential function.) The following observation indicates a relation between  $H(K; u)$  and the Alexander polynomial.

**Observation 4.3.** *Let  $\Delta(K; t)$  be the Alexander polynomial of a knot  $K$ . For the figure-eight knot and torus knots, the equations  $H(K; u) = 0$  and  $\Delta(K; \exp(u)) = 0$  share a root. Note that here we regard  $H(K; u)$  as a function defined on the whole complex plane.*

*Proof.* First note that  $\Delta(E; t) = -t^2 + 3t - 1$  and  $\Delta(T(a, b); t) = (t^{ab} - 1)(t - 1)/(t^a - 1)(t^b - 1)$ .

For the figure-eight knot  $E$ , we have from [15]

$$H(E; u) = \text{Li}_2(y^{-1} \exp(-u)) - \text{Li}_2(y \exp(-u)) + (\log(-y) + \pi \sqrt{-1}) u,$$

where  $y$  is a solution to the equation  $y + y^{-1} = 2 \cosh(u) - 1$ . Put  $u := \pm \text{arccosh}(3/2)$ . Then  $y = 1$  and so  $H(E; u)$  vanishes. Note that we take the branch of  $\log$  so that  $\log(-1) = -\pi \sqrt{-1}$ . It is easy to see that  $\exp(\pm \text{arccosh}(3/2))$  are the roots of  $\Delta(E; \exp(u)) = 0$ .

For the torus knot  $T(a, b)$  the zero of  $H(T(a, b); u)$  is  $2\pi \sqrt{-1}/(ab) - 2\pi \sqrt{-1}$ . It is also a zero of  $\Delta(T(a, b); \exp(u))$ .  $\square$

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